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On the Stability of Multivariate Trigonometric Systems*

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Kadec's 1/4-theorem says that if $\{\lambda_n: n \in \mathbf{Z}\}$ is a sequence of real numbers for which $|\lambda_n - n| \le L < \frac{1}{4}$, then $\{e^{i\lambda_n \omega}: n \in \mathbf{Z}\}$ forms a Riesz basis for $L^2[-\pi, \pi]$. S. Favier, R. Zalik, C. Chui, and X. Shi extended this result to the multivariate case. But their results lead to very small stability bounds. In this paper, we give an optimal stability bound for the multivariate trigonometric systems. Moreover, for the case of Fourier frames in $L^2[-\pi, \pi]^d$, we also give the stability bounds.

Key Words: Kadec's 1/4-theorem; stability bound; frame; Riesz basis.

1. INTRODUCTION AND MAIN RESULTS

A family of functions $\{f_j: j \in J\}$ belonging to a separable Hilbert space $\mathscr{H}$ is said to be a frame if there exist positive constants A and B such that

$$A\|f\|^2 \le \sum_{j \in J} |\langle f, f_j \rangle|^2 \le B\|f\|^2$$

for every $f \in \mathscr{H}$. The numbers A and B are called the lower and upper frame bounds, respectively. If only the right-hand inequality is satisfied for every $f \in \mathscr{H}$, then $\{f_j: j \in J\}$ is said to be a Bessel sequence with bound B.

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Family $\{f_j: j \in J\}$ is called a Riesz basis if it is complete in $\mathscr{H}$ and there exist positive constants A and B such that

$$A \sum_{j \in J} |c_j|^2 \le \left\| \sum_{j \in J} c_j f_j \right\|^2 \le B \sum_{j \in J} |c_j|^2, \qquad \forall \{c_j\} \in \ell^2(J).$$

A frame that ceases to be a frame when any one of its elements is removed is said to be an exact frame. It is well known that exact frames and Riesz basis are identical (see [7]).

Kadec's $1/4$-theorem [6] states that if $\{\lambda_n\}$ is a sequence of real numbers for which

$$|\lambda_n - n| \le L < \frac{1}{4}, \qquad n \in \mathbf{Z},$$

then $\{e^{i\lambda_n \omega}\}$ forms a Riesz basis for $L^2[-\pi, \pi]$.

For the multivariate case, we can ask a similar question. Let $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbf{Z}^d$, $\lambda_{\mathbf{n}} = (\lambda_{n_1}, \ldots, \lambda_{n_d}) \in \mathbf{R}^d$. We want to find a constant θ_d such that $\{e^{i \langle \lambda_{\mathbf{n}}, \omega \rangle}: \mathbf{n} \in \mathbf{Z}^d\}$ is a Riesz basis for $L^2[-\pi, \pi]^d$ whenever

$$\sup_{\mathbf{n} \in \mathbf{Z}^d} \|\lambda_{\mathbf{n}} - \mathbf{n}\|_\infty = \sup_{\mathbf{n} \in \mathbf{Z}^d} \sup_{1 \le k \le d} |\lambda_{n_k} - n_k| < \theta_d.$$

We call θ_d a stability bound.

Favier and Zalik proved the following proposition [5, Corollary 2].

Proposition 1.1. *Assume that* $|n_k - \lambda_{n_k}| \le L$, $k = 1, \ldots, d$, *and that* $L < \frac{1}{4}$. *If* $B_d(L) < 1$, *then* $\{e^{i \langle \lambda_{\mathbf{n}}, \omega \rangle}: \mathbf{n} \in \mathbf{Z}^d\}$ *is a Riesz basis in* $L^2[-\pi, \pi]^d$ *with frame bounds* $[1 - B_d(L)^{1/2}]^2$ *and* $[1 + B_d(L)^{1/2}]^2$, *where* $B_d(L)$ *is defined recursively as follows*: $B_1(L) = 1 - \cos \pi L + \sin \pi L$, *and for* $d > 1$, $B_d(L) := \{B_{d-1}^{1/2}(L) + B_1^{1/2}(L)[1 + B_{d-1}^{1/2}(L)]\}^2$.

In [4], Chui and Shi improved this proposition. They proved the following.

Proposition 1.2 ([4, Theorem 2]). *The* (*unique*) *zero* θ_d *of* $B_d^*(t) - 1$ *in* $[0, \frac{1}{4}]$ *is a stability bound, where*

$$B_d^*(t) := \frac{1 - \cos \pi t + \sin \pi t}{H(t)} \left[(H(t) + 1)^d - 1 \right],$$

$$H(t) := \frac{\sin \pi t}{\pi t} - \cos \pi t + \sin \pi t.$$

The stability bounds obtained in Proposition 1.1 and Proposition 1.2 are as follows [4]:

TABLE I
Stability Bounds θ_d

d	Favier–Zalik	Chui–Shi	d	Favier–Zalik	Chui–Shi
1	0.25	0.25	5	0.00696	0.04458
2	0.05077	0.11481	6	0.00473	0.03706
3	0.02083	0.07515	7	0.00342	0.03172
4	0.01119	0.05594	8	0.00259	0.02773

It is easy to see that both the Favier–Zalik bounds and the Chui–Shi bounds are very small for large d. In this paper, we show that $\theta_d = \frac{1}{4}$ is a stability bound for any $d \geq 1$ and that $\frac{1}{4}$ cannot be improved. In fact, we prove the following.

THEOREM 1.1. *Let $\{\lambda_{\mathbf{n}}: \mathbf{n} \in \mathbf{Z}^d\}$ be a sequence in $\mathbf{R}^d$ for which*

$$L := \sup_{\mathbf{n} \in \mathbf{Z}^d} \|\lambda_{\mathbf{n}} - \mathbf{n}\|_\infty < \frac{1}{4}; \tag{1}$$

then $\{e^{i\langle \lambda_{\mathbf{n}}, \omega\rangle}: \mathbf{n} \in \mathbf{Z}^d\}$ forms a Riesz basis for $L^2[-\pi, \pi]^d$ with frame bounds $(2\pi)^d[1 - B(L)]^{2d}$ and $(2\pi)^d[1 + B(L)]^{2d}$, where $B(L) = 1 - \cos \pi L + \sin \pi L$. Moreover, the inequality in (1) cannot be replaced by equality.

We call $\{e^{i\langle \lambda_{\mathbf{n}}, \omega\rangle}: \mathbf{n} \in \mathbf{Z}^d\}$ a Fourier frame, if it constitutes a frame for $L^2[-\pi, \pi]^d$. For the case of one dimension, Balan [1] and Christensen [3] gave the stability bounds of Fourier frames. Here we give a multivariate version.

THEOREM 1.2. *Let $\mu_{\mathbf{n}} = (\mu_{n_1}, \dots, \mu_{n_d})$ and $\lambda_{\mathbf{n}} = (\lambda_{n_1}, \dots, \lambda_{n_d})$ for $\mathbf{n} \in \mathbf{Z}^d$. If $\{e^{i\langle \mu_{\mathbf{n}}, \omega\rangle}: \mathbf{n} \in \mathbf{Z}^d\}$ is a frame for $L^2[-\pi, \pi]^d$ with bounds A and B and*

$$L := \sup_{\mathbf{n} \in \mathbf{Z}^d} \|\lambda_{\mathbf{n}} - \mu_{\mathbf{n}}\|_\infty < \frac{1}{4} - \frac{1}{\pi}\arcsin\left(\frac{1 - \sqrt{A/B}}{\sqrt{2}}\right),$$

then $\{e^{i\langle \lambda_{\mathbf{n}}, \omega\rangle}: \mathbf{n} \in \mathbf{Z}^d\}$ is a frame for $L^2[-\pi, \pi]^d$ with bounds $A[1 - \sqrt{B/A}\, B(L)]^{2d}$ and $B[1 + B(L)]^{2d}$.

The proofs of the previous two theorems as well as the proofs of Favier–Zalik and Chui–Shi use the following fact: if one of the components of $\mathbf{n} = (n_1, \dots, n_d)$ changes, then only the corresponding component of $\lambda_{\mathbf{n}}$ changes. In other words, these proofs are valid only when $\{e^{i\langle \lambda_{\mathbf{n}}, \omega\rangle}: \mathbf{n} \in \mathbf{Z}^d\} = \{e^{i\lambda_{n_1}\omega_1} \cdots e^{i\lambda_{n_d}\omega_d}: n_1, \dots, n_d \in \mathbf{Z}\}$.

In general, when one of the components of $\mathbf{n} = (n_1, \ldots, n_d)$ changes, all of the components of $\lambda_{\mathbf{n}}$ may change. For this case, we have

THEOREM 1.3. *Let* $\alpha_{\mathbf{n}} = (\alpha_{\mathbf{n}}^1, \ldots, \alpha_{\mathbf{n}}^d)$ *and* $\beta_{\mathbf{n}} = (\beta_{\mathbf{n}}^1, \ldots, \beta_{\mathbf{n}}^d)$, $\mathbf{n} \in \mathbf{Z}^d$. *Suppose that* $\{e^{i\langle \beta_{\mathbf{n}}, \omega\rangle}: \mathbf{n} \in \mathbf{Z}^d\}$ *is a frame for* $L^2[-\pi, \pi]^d$ *with bounds* A *and* B. *If*

$$D(L) := \left(1 - \cos \pi L + \sin \pi L + \frac{\sin \pi L}{\pi L}\right)^d - \left(\frac{\sin \pi L}{\pi L}\right)^d < \sqrt{\frac{A}{B}}, \qquad 0 \le L < \tfrac{1}{4}$$

and

$$\|\beta_{\mathbf{n}} - \alpha_{\mathbf{n}}\|_\infty \le L, \qquad \mathbf{n} \in \mathbf{Z}^d;$$

then $\{e^{i\langle \alpha_{\mathbf{n}}, \omega\rangle}: \mathbf{n} \in \mathbf{Z}^d\}$ *is a frame for* $L^2[-\pi, \pi]^d$ *with bounds* $A[1 - \sqrt{B/A}\,D(L)]^2$ *and* $B[1 + D(L)]^2$.

In particular, if $\beta_{\mathbf{n}} = \mathbf{n}$, $0 \le L < \frac{1}{4}$, $D(L) < 1$ *and*

$$\|\alpha_{\mathbf{n}} - \mathbf{n}\|_\infty \le L, \qquad \mathbf{n} \in \mathbf{Z}^d,$$

then $\{e^{i\langle \alpha_{\mathbf{n}}, \omega\rangle}: \mathbf{n} \in \mathbf{Z}^d\}$ *is a Riesz basis for* $L^2[-\pi, \pi]^d$ *with bounds* $(2\pi)^d [1 - D(L)]^2$ *and* $(2\pi)^d \cdot [1 + D(L)]^2$.

Notation. In this paper, the norms of all Hilbert spaces are denoted by $\|\cdot\|$. The exact meaning can be seen by context.

2. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. We will prove the theorem for $d = 2$. If $d > 2$, the proof is similar.

By Kadec's $\frac{1}{4}$-theorem, both $\{e^{i\lambda_{n_1}\omega_1}: n_1 \in \mathbf{Z}\}$ and $\{e^{i\lambda_{n_2}\omega_2}: n_2 \in \mathbf{Z}\}$ are Riesz bases for $L^2[-\pi, \pi]$ with bounds $2\pi[1 - B(L)]^2$ and $2\pi[1 + B(L)]^2$. Hence for any finite sequence of complex numbers $\{c_{n_1, n_2}: n_1, n_2 \in \mathbf{Z}\}$, we have

$$\begin{aligned}\Big\| \sum_{n_1, n_2} c_{n_1, n_2} e^{i(\lambda_{n_1}\omega_1 + \lambda_{n_2}\omega_2)} \Big\|^2 &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Big| \sum_{n_1} \Big(\sum_{n_2} c_{n_1, n_2} e^{i\lambda_{n_2}\omega_2} \Big) e^{i\lambda_{n_1}\omega_1} \Big|^2 \, d\omega_1 \, d\omega_2 \\ &\le \int_{-\pi}^{\pi} 2\pi [1 + B(L)]^2 \sum_{n_1} \Big| \sum_{n_2} c_{n_1, n_2} e^{i\lambda_{n_2}\omega_2} \Big|^2 \, d\omega_2 \\ &= 2\pi [1 + B(L)]^2 \sum_{n_1} \int_{-\pi}^{\pi} \Big| \sum_{n_2} c_{n_1, n_2} e^{i\lambda_{n_2}\omega_2} \Big|^2 \, d\omega_2 \\ &\le (2\pi)^2 [1 + B(L)]^4 \sum_{n_1} \sum_{n_2} |c_{n_1, n_2}|^2 .\end{aligned}$$

A similar argument shows that

$$\Big\| \sum_{n_1, n_2} c_{n_1, n_2} e^{i(\lambda_{n_1}\omega_1 + \lambda_{n_2}\omega_2)} \Big\|^2 \ge (2\pi)^2 [1 - B(L)]^4 \sum_{n_1} \sum_{n_2} |c_{n_1, n_2}|^2 .$$

On the other hand, since both $\{e^{i\lambda_{n_1}\omega_1}: n_1 \in \mathbf{Z}\}$ and $\{e^{i\lambda_{n_2}\omega_2}: n_2 \in \mathbf{Z}\}$ are complete in $L^2[-\pi, \pi]$, $\{e^{i(\lambda_{n_1}\omega_1 + \lambda_{n_2}\omega_2)}: n_1, n_2 \in \mathbf{Z}\}$ is complete in $L^2[-\pi, \pi]^2$. By [7, Theorem 1.9] we know that $\{e^{i(\lambda_{n_1}\omega_1 + \lambda_{n_2}\omega_2)}: n_1, n_2 \in \mathbf{Z}\}$ is a Riesz basis for $L^2[-\pi, \pi]^2$ with bounds $(2\pi)^2[1 - B(L)]^4$ and $(2\pi)^2 \cdot [1 + B(L)]^4$.

Moreover, the counterexamples in [7, pp. 122–125] can also be extended to the case of a multivariate, so the inequality (1) cannot be replaced by equality. In fact, an explicit counterexample was shown in [4].

To prove Theorem 1.2, we need the following lemma.

LEMMA 2.1. *Suppose* $\mu_{\mathbf{n}} = (\mu_{n_1}, \dots, \mu_{n_d})$ *for* $\mathbf{n} \in \mathbf{Z}^d$. *Then* $\{e^{i\langle \mu_{\mathbf{n}}, \omega\rangle}: \mathbf{n} \in \mathbf{Z}^d\}$ *is a frame* (*Riesz basis*) *for* $L^2[-\pi, \pi]^d$ *if and only if* $\{e^{i\mu_{n_k}\omega_k}: n_k \in \mathbf{Z}\}$ *is also a frame* (*Riesz basis*) *for* $L^2[-\pi, \pi]$ *for any* $1 \le k \le d$.

Moreover, if the conditions are satisfied and A_k *and* B_k *are the frame bounds for* $\{e^{i\mu_{n_k}\omega_k}: n_k \in \mathbf{Z}\}$, *then* $A = A_1 A_2 \cdots A_d$ *and* $B = B_1 B_2 \cdots B_d$.

Proof. Again, we consider the case $d = 2$. If $d > 2$, the proof is similar. First, we show the necessity. Without loss of generality, we take $k = 1$.

Let $\{e^{i\langle \mu_{\mathbf{n}}, \omega\rangle}: \mathbf{n} \in \mathbf{Z}^d\}$ be a frame for $L^2[-\pi, \pi]^d$. Fix some $g \in L^2[-\pi, \pi]$ such that $\|g\| > 0$. For any $f \in L^2[-\pi, \pi]$, we have

$$A\|f(\omega_1)g(\omega_2)\|^2 \le \sum_{n_1, n_2} \left|\left\langle f(\omega_1)g(\omega_2), e^{i(\mu_{n_1}\omega_1 + \mu_{n_2}\omega_2)}\right\rangle\right|^2$$

$$\le B\|f(\omega_1)g(\omega_2)\|^2. \tag{2}$$

Since $\|f(\omega_1)g(\omega_2)\| = \|f\| \cdot \|g\|$, the above inequality implies

$$A\|f\|^2\|g\|^2 \le \sum_{n_1}\left|\left\langle f(\omega_1), e^{i\mu_{n_1}\omega_1}\right\rangle\right|^2 \sum_{n_2}\left|\left\langle g(\omega_2), e^{i\mu_{n_2}\omega_2}\right\rangle\right|^2 \le B\|f\|^2\|g\|^2.$$

Let $D = \|g\|^2/\sum_{n_2}|\langle g(\omega_2), e^{i\mu_{n_2}\omega_2}\rangle|^2$. Then $0 < D < +\infty$ and

$$AD\|f\|^2 \le \sum_{n_1}\left|\left\langle f(\omega_1), e^{i\mu_{n_1}\omega_1}\right\rangle\right|^2 \le BD\|f\|^2, \qquad \forall f \in L^2[-\pi, \pi].$$

Hence $\{e^{i\mu_{n_1}\omega_1}: n_1 \in \mathbf{Z}\}$ is a frame for $L^2[-\pi, \pi]$.

For the case of Riesz basis, since a Riesz basis is also a frame, we only need to show that $\{e^{i\mu_{n_1}\omega_1}\}$ is linearly independent.

Fix some finitely nonzero complex sequence $\{d_{n_2}: n_2 \in \mathbf{Z}\}$ such that $\sum_{n_2}|d_{n_2}|^2 > 0$. For any finite complex sequence $\{c_{n_1}: n_1 \in \mathbf{Z}\}$, a similar argument shows that

$$AD' \sum_{n_1}|c_{n_1}|^2 \le \left\|\sum_{n_1} c_{n_1} e^{i\mu_{n_1}\omega_1}\right\|^2 \le BD' \sum_{n_1}|c_{n_1}|^2,$$

where $D' = \sum_{n_2}|d_{n_2}|^2/\|\sum_{n_2} d_{n_2} e^{i\mu_{n_2}\omega_2}\|^2$. This implies $\{e^{i\mu_{n_1}\omega_1}\}$ is linearly independent in $L^2[-\pi, \pi]$.

Next, we show the sufficiency. Let $\{e^{i\mu_{n_k}\omega_k}: n_k \in \mathbf{Z}\}$ be a frame for $L^2[-\pi, \pi]$ with bounds A_k and B_k, $k = 1, 2$. For any $f \in L^2[-\pi, \pi]^2$, we have

$$\sum_{n_1, n_2 \in \mathbf{Z}} \left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\omega_1, \omega_2) e^{-i(\mu_{n_1}\omega_1 + \mu_{n_2}\omega_2)}\, d\omega_1\, d\omega_2\right|^2$$

$$= \sum_{n_2}\sum_{n_1}\left|\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(\omega_1, \omega_2) e^{-i\mu_{n_2}\omega_2}\, d\omega_2\right) e^{-i\mu_{n_1}\omega_1}\, d\omega_1\right|^2$$

$$\le \sum_{n_2} B_1 \int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi} f(\omega_1, \omega_2) e^{-i\mu_{n_2}\omega_2}\, d\omega_2\right|^2 d\omega_1$$

$$= B_1 \int_{-\pi}^{\pi} \sum_{n_2} \left| \int_{-\pi}^{\pi} f(\omega_1, \omega_2) e^{-i\mu_{n_2}\omega_2}\, d\omega_2 \right|^2 d\omega_1$$

$$\le B_1 B_2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(\omega_1, \omega_2)|^2\, d\omega_2\, d\omega_1. \tag{3}$$

A similar argument shows that

$$\sum_{n_1, n_2 \in \mathbf{Z}} \left| \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\omega_1, \omega_2) e^{-i(\mu_{n_1}\omega_1 + \mu_{n_2}\omega_2)}\, d\omega_1\, d\omega_2 \right|^2$$

$$\ge A_1 A_2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(\omega_1, \omega_2)|^2\, d\omega_2\, d\omega_1. \tag{4}$$

Hence $\{e^{i\langle \mu_{\mathbf{n}}, \omega\rangle}: \mathbf{n} \in \mathbf{Z}^2\}$ is a frame for $L^2[-\pi, \pi]^2$. For the case of Riesz bases, the proof is similar to that of Theorem 1.1.

Moreover, suppose that A_k and B_k are the frame bounds for $\{e^{i\mu_{n_k}\omega_k}: n_k \in \mathbf{Z}\}$, $k = 1, 2$. By (2), (3), and (4), it is easy to see that $A = A_1 A_2$ and $B = B_1 B_2$.

Proof of Theorem 1.2. For any $1 \le k \le d$, by Lemma 2.1, $\{e^{i\mu_{n_k}\omega_k}: n_k \in \mathbf{Z}\}$ is a frame for $L^2[-\pi, \pi]$ with bounds A_k and B_k for which $A = A_1 A_2 \cdots A_d$ and $B = B_1 B_2 \cdots B_d$. Since $B_k/A_k \ge 1$, we have $B_k/A_k \le B/A$. Hence

$$\frac{1}{4} - \frac{1}{\pi} \arcsin \frac{1 - \sqrt{A_k/B_k}}{\sqrt{2}} \ge \frac{1}{4} - \frac{1}{\pi} \arcsin \frac{1 - \sqrt{A/B}}{\sqrt{2}}.$$

By [1, Theorem 1], $\{e^{i\lambda_{n_k}\omega_k}: n_k \in \mathbf{Z}\}$ is a frame for $L^2[-\pi, \pi]$ with bounds $A_k[1 - \sqrt{\frac{B}{A}} B(L)]^2$ and $B_k[1 + B(L)]^2$. The conclusion follows by Lemma 2.1.

Proof of Theorem 1.3. Let

$$a_0(\delta) = \frac{\sin \pi\delta}{\pi\delta}, \qquad a_k(\delta) = \frac{(-1)^{k+1} 2\delta \sin \pi\delta}{\pi(k^2 - \delta^2)},$$

$$a_{-k}(\delta) = \frac{i(-1)^{k+1} 2\delta \cos \pi\delta}{\pi\left[\left(k - \frac{1}{2}\right)^2 - \delta^2\right]},$$

$$\varphi_0(x) = 1, \qquad \varphi_k(x) = \cos kx,$$

$$\varphi_{-k}(x) = \sin\left(\left(k - \tfrac{1}{2}\right)x\right), \qquad k \ge 1.$$

For any $k \neq 0$, $|a_k(\delta)|$ is increasing for $\delta \in [0, \frac{1}{4}]$ and $\sum_{k \neq 0} |a_k(L)| = 1 - \cos \pi L + \sin \pi L + (\sin \pi L)/\pi L - 1$ (see [1, 7]). Moreover,

$$1 - e^{i\delta x} = 1 - \sum_{k \in \mathbf{Z}} a_k(\delta)\varphi_k(x), \qquad x \in [-\pi, \pi].$$

Hence

$$1 - e^{i(\delta_1\omega_1 + \cdots + \delta_d\omega_d)} = 1 - \sum_{\mathbf{k} \in \mathbf{Z}^d} \prod_{q=1}^{d} \left[a_{k_q}(\delta_q)\varphi_{k_q}(\omega_1) \right],$$

$$\mathbf{k} = (k_1, \ldots, k_d).$$

For any finitely nonzero sequence of complex numbers $\{c_{\mathbf{n}}\}$, we have

$$\begin{aligned}
&\left\| \sum_{\mathbf{n} \in \mathbf{Z}^d} c_{\mathbf{n}} \left(e^{i\langle \beta_{\mathbf{n}}, \omega \rangle} - e^{i\langle \alpha_{\mathbf{n}}, \omega \rangle} \right) \right\| \\
&\quad = \left\| \sum_{\mathbf{n} \in \mathbf{Z}^d} c_{\mathbf{n}} e^{i\langle \beta_{\mathbf{n}}, \omega \rangle} \left(1 - e^{i\langle \delta_{\mathbf{n}}, \omega \rangle} \right) \right\| \\
&\qquad \left(\delta_{\mathbf{n}} = \alpha_{\mathbf{n}} - \beta_{\mathbf{n}} = \left(\delta_{\mathbf{n}}^1, \ldots, \delta_{\mathbf{n}}^d \right) \right) \\
&\quad = \left\| \sum_{\mathbf{n} \in \mathbf{Z}^d} c_{\mathbf{n}} e^{i\langle \beta_{\mathbf{n}}, \omega \rangle} \left(1 - \sum_{\mathbf{k} \in \mathbf{Z}^d} \prod_{q=1}^{d} a_{k_q}(\delta_{\mathbf{n}}^q)\varphi_{k_q}(\omega_q) \right) \right\| \\
&\quad \leq \left\| \sum_{\mathbf{n} \in \mathbf{Z}^d} c_{\mathbf{n}} e^{i\langle \beta_{\mathbf{n}}, \omega \rangle} \left(1 - \prod_{q=1}^{d} a_0(\delta_{\mathbf{n}}^q) \right) \right\| \\
&\qquad + \sum_{\mathbf{k} \neq \mathbf{0}} \left\| \left[\prod_{q=1}^{d} \varphi_{k_q}(\omega_q) \right] \sum_{\mathbf{n}} c_{\mathbf{n}} e^{i\langle \beta_{\mathbf{n}}, \omega \rangle} \prod_{q=1}^{d} a_{k_q}(\delta_{\mathbf{n}}^q) \right\| \\
&\quad \leq \left(1 - \left(\frac{\sin \pi L}{\pi L} \right)^d \right) \sqrt{B} \left(\sum_{\mathbf{n}} |c_{\mathbf{n}}|^2 \right)^{1/2} \\
&\qquad + \sum_{\mathbf{k} \neq \mathbf{0}} \sqrt{B} \left(\sum_{\mathbf{n}} \left| c_{\mathbf{n}} \prod_{q=1}^{d} a_{k_q}(\delta_{\mathbf{n}}^q) \right|^2 \right)^{1/2} \\
&\quad \triangleq P + Q,
\end{aligned} \tag{5}$$

where letting $G_p = \{\mathbf{k} \in \mathbf{Z}^d\colon p \text{ components of } \mathbf{k} \text{ are } 0\}$, $1 \leq p \leq d - 1$ we have:

$$Q = \sum_{p=0}^{d-1} \sum_{\mathbf{k} \in G_p} \sqrt{B} \left(\sum_{\mathbf{n}} \left| c_{\mathbf{n}} \prod_{q=1}^{d} a_{k_q}(\delta_{\mathbf{n}}^q) \right|^2 \right)^{1/2}$$

$$\leq \sum_{p=0}^{d-1} \binom{d}{p} \sum_{k_{p+1}, \ldots, k_d \neq 0} \prod_{q=p+1}^{d} \left| a_{k_q}(L) \right| \sqrt{B} \left(\sum_{\mathbf{n}} |c_{\mathbf{n}}|^2 \right)^{1/2}$$

$$= \sum_{p=0}^{d-1} \binom{d}{p} \left(\sum_{k_1 \neq 0} \left| a_{k_1}(L) \right| \right)^{d-p} \sqrt{B} \left(\sum_{\mathbf{n}} |c_{\mathbf{n}}|^2 \right)^{1/2}$$

$$= \left(\left(1 - \cos \pi L + \sin \pi L + \frac{\sin \pi L}{\pi L} \right)^d - 1 \right) \sqrt{B} \left(\sum_{\mathbf{n}} |c_{\mathbf{n}}|^2 \right)^{1/2}. \quad (6)$$

By (5) and (6), $\{e^{i\langle \beta_{\mathbf{n}}, \omega \rangle} - e^{i\langle \alpha_{\mathbf{n}}, \omega \rangle}\}$ is a Bessel sequence with bound $(D(L))^2 B$. By [2, Theorem 1], $\{e^{i\langle \alpha_{\mathbf{n}}, \omega \rangle} : \mathbf{n} \in \mathbf{Z}^d\}$ is a frame for $L^2[-\pi, \pi]^d$ when $D(L) < \sqrt{A/B}$. This completes the proof.

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